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DESIGN OF MULTIPLE-VEHICLES DELIVERY TOURS SATISFYING DURATION CONSTRAINTS

by

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ABSTRACT

We have studied the following organization for the letter and parcel pick-up and delivery problem in urban environment: the day is divided in periods of time during which each vehicle starts from a sorting center, travels to a very precise area of the region, picks up and delivers letters and parcels and returns to the depot to have the collected material sorted out for delivery during the following period. The intent of this approach is to have each zone serviced by one vehicle only in order to facilitate (1) the work of the dispatcher and (2) the routing of each vehicle. We have developed a method of partitioning an urban region into zones to be assigned, each one to a specific vehicle, so as to minimize the total number of vehicles used or the total distance traveled by all the vehicles. The first part of the method determines the zones and the average number of points to visit. Then, a first refinement of the method takes into account the daily variability of the demand in each zone, whereas a second refinement examines the advantage of allowing some overlapping of zones for the purpose of having an overloaded vehicle relieved by one from an adjacent zone.

RÉSUMÉ

Nous avons étudié l'organisation suivante pour le problème de la cueillette et la livraison de lettres et colis en milieu urbain : la journée est divisée en tranches de temps durant lesquelles chaque véhicule part d'un centre de tri, se rend dans un secteur très précis de la région pour y effectuer la cueillette et la livraison de colis, puis revient au centre pour faire trier les colis ramassés pour livraison dans la tranche de temps suivante. Le but visé par cette approche est d'allouer une zone à chaque véhicule de façon à faciliter 1. la tâche du répartiteur et 2. la confection des itinéraires de chacun des véhicules. Nous avons mis au point une méthode qui permet de séparer une région urbaine en zones affectées, chacune, à un seul véhicule, de façon à minimiser le nombre total de véhicules utilisés ou la distance totale parcourue par l'ensemble des véhicules. La première partie de la méthode permet de déterminer chacune des zones et le nombre moyen de points à visiter dans chacune de celles-ci. Un premier raffinement de la méthode permet de tenir compte de la variabilité journalière de la demande dans chacune des zones. Un deuxième raffinement permet de déterminer les zones lorsqu'on permet un certain chevauchement de celles-ci de telle façon qu'un véhicule surchargé puisse être soulagé par un véhicule d'une zone adjacente.

1. Introduction

The optimal routing for the multi-vehicle pick-up and delivery problem is a complex one to solve because of its highly combinatoric nature: the exponential increase of the possible solutions in relation to the number of points to be visited precludes the design of an optimal and general solution.

A possible approach consists in dividing the territory to cover into zones assigned to each vehicle, which reduces the size of the routing problems to be resolved. The present paper studies the way one can determine the zones in such a way as to obtain the greatest efficiency of transportation.

In the problem under consideration, each vehicle, in a fixed period of time, starts from a central depot (where the sorting center is located), travels to its assigned zone to do the pick-up and delivery of the letters and parcels (delivers whatever was picked up during the previous period) and returns to the depot for the sorting of the material picked up. Due to the

small number of letters and parcels that a vehicle can collect in a given period, no constraints of capacity on the vehicles are considered.

The model applies to the transportation of letters and small parcels in urban area, market which has had a terrific growth in the past five years in North America. The form of organization considered in this paper becomes advantageous when the volume of demand and the number of vehicles are large enough to justify the utilization of a sorting center.

For the sake of analysis, we are considering a circular area with the depot located in its center and the points to be visited distributed at random with a density gradually decreasing from the center toward the periphery. The points to be visited are located on a set of roads forming a fine rectangular or ring-radial grid.

The purpose of the following analysis is to determine the shape and orientation of the zones so as to minimize the number of vehicles or the total distance traveled by the set of vehicles.

The model will also permit to compare the following alternatives: offer a pick-up service with delivery the next day, the same day or within a few hours. Finally, some refinements of the model are taking care of the daily variation of the demand in each zone and of the possible zone overlapping in order to reduce the impacts of such variation.

Newell and Daganzo[1986] have studied the zone designing problem with vehicles having limits of capacity. Nevertheless, their model does not apply to the problem considered here since there is no direct relation between the limit of time and the limit of capacity because of the line-haul portion of the trips. On the other hand, the first part of our analysis is similar in scope to the study of Newell and Daganzo.

In section 2, we summarize some results from the literature which support the first part of our development. In section 3, we describe the principal method of dividing a region into zones, each one being assigned to just one vehicle. In section 4, we reconsider this partitioning of zones taking into account the daily variability in the demand and we examine the possibility of overlapping zones in order to reduce the variance of the demand and hence, the number of vehicles required for an adequate coverage of the region.

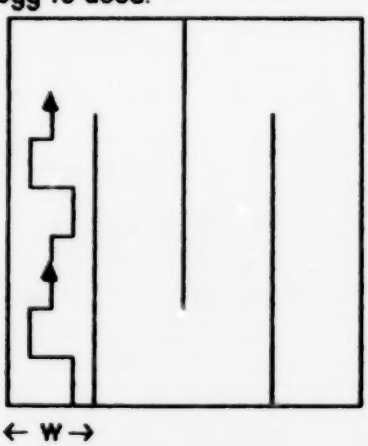
2. Literature review

In order to adequately plan the zones for the design of the routing, it is necessary to estimate the length of a tour in relation to the area (and the shape) of the zone. Larson and Odoni [1981] present the only one known approximation formula: length D of an optimal tour for n points randomly and uniformly distributed over a zone of area A is given by

$$D = k\sqrt{An}$$

This relation applies to Euclidean and L_1 metrics. The k value has been estimated at 0.765 (Stein[1978]) for the Euclidean case and is not known for the L_1 case. This relation is however only valid when $n \geq 15$ and the area sufficiently compact and convex. It has not been demonstrated that this formula remained valid for regions of elongated shape. One can assume that for elongated regions of same area, the length of tour might be longer. So that the approximation formula is not usable in our context where the appropriate shape of the zones has to be determined to minimize travel times.

Daganzo[1984a] developed an approximate formula to evaluate the length of a tour based on a sub-optimal strategy; this method, which gives a length slightly greater than that of the optimal tour in the case of a compact zone has the advantage of being applicable also to the narrow and elongated zones, unlike the previous formula. The distance traveled using this strategy is estimated at $0.9\sqrt{An}$ in the Euclidean case and at $1.15\sqrt{An}$ in the L_1 case. It can be expected that this strategy will give a solution close to the optimum for narrow regions. Furthermore, the routing is greatly facilitated if this strategy is used.



The strategy consists in decomposing a zone to be covered into a strip of width w and to visit the points in the longitudinal order within the strip.

Daganzo[1984a] has determined that the optimal width of the strip for a L_1 metric was:

$$w_0 = (3/\partial)^{1/2}$$

where ∂ is the density (number of points per unit of area).

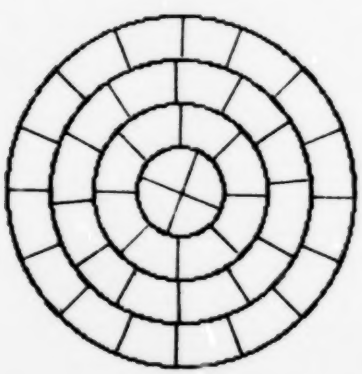
Newell and Daganzo[1986] have shown that in the case of multiple vehicles with capacity constraint, the zones covered by each one of the vehicles had to have an approximately rectangular shape elongated toward the source and that the set of zones had to be arranged in concentric rings.

They further demonstrated that the optimal width of a strip in a rectangular zone remained valid in a sector of a concentric ring (at least when the sector is approximately rectangular). Also, in order to minimize the line-haul time, the strip should start and end on the side of the sector nearest to the source. Furthermore, each zone is divided into two strips in such a way that the vehicle only makes one round trip inside of the zone.

One can find in the literature a few other works on approximate analytical methods of strategic planning in relation to the distribution of the demand: Burns and al.[1985], Daganzo[1978,1984b,c, 1985a,b,c,], Daganzo and Newell[1986], Newell and Daganzo[1986b,c] and Han[1984]. These works are dealing with problems slightly different than that considered in this paper and, consequently, will not be discussed here.

3. Design of zones

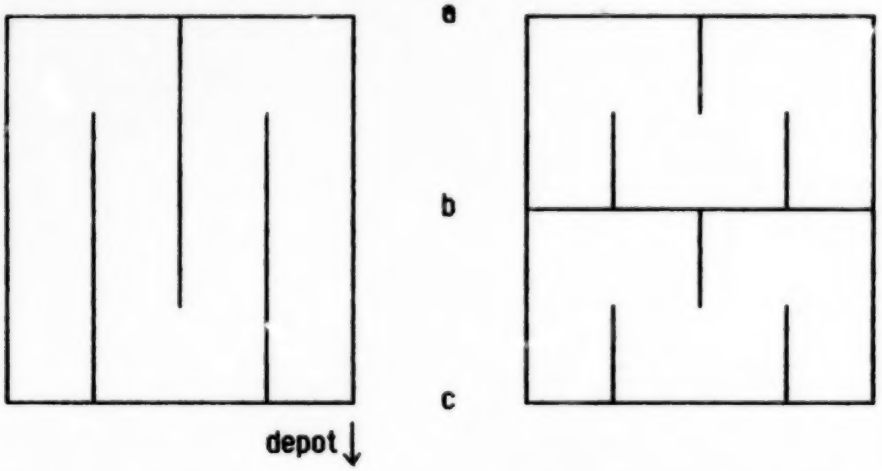
If we partition a circular region into zones approximately rectangular and elongated toward the center, we obtain a set of zones arranged in concentric rings:



Let us examine how we have to cut the zones e.g. specify the internal and external radii of each ring and the number N_x of zones per ring.

3.1. Formulation

We will consider that each sector is approximately 2w wide so that one vehicle does only one round trip in it from the point closest to the source. Starting and ending the pick-up and delivery tour of a zone on the side nearest to the center minimizes the line-haul distance. Furthermore, using zones of width 2w in order to make only one round trip is justified by the loss of efficiency caused by the increase of the line-haul distance if one selects zones of width 2iw with i>1 and integer. Let us consider, for example, the solutions of two decompositions of a region, one into two zones of width 2w and the other into two zones 4w wide: the four zones thus formed are of equal area and may be serviced in the same time by a vehicle



solution 1: 2 zones of width 2w solution 2: 2 zones of width 4w
(a, b and c are the distances measured from the depot)

Solution 1 gives a total distance of:

$$D_1 = 2k\sqrt{An} + 4a$$

where A is the area of one zone and n the number of points to be visited.
Solution 2 gives a total distance of:

$$D_2 = 2k\sqrt{An} + 2a + 2b.$$

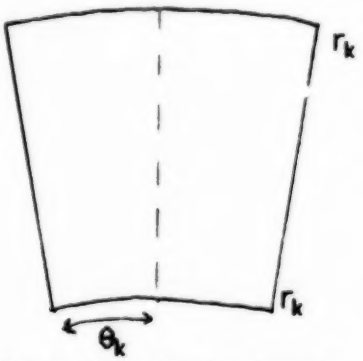
We thus have $D_2 - D_1 = 2(b-a) > 0$. Solution 1 is then preferable.

To evaluate the total distance traveled by the set of vehicles we will separate the distance traveled by the vehicle in radial distance (including the line-haul distance from the depot) and the transverse distance.

If the point density varies as a function of the radius ($\partial(r)$), then the number of points in the k^{th} ring of internal radius r_{k-1} and external radius r_k is given by:

$$\int_{r_{k-1}}^{r_k} 2\Pi\partial(r)rdr$$

For a sector of this ring (each sector corresponds to the zone assigned to only one vehicle), θ_k corresponds to half the angle.



The total transverse distance in a ring is given by:

$$\int_{r_{k-1}}^{r_k} 2\Pi\partial(r)\theta_k r^2 dr/3 \qquad \text{with } \theta_k = \Pi/N_k$$

where N_k is the number of vehicles (or sectors) in the k^{th} ring.

The total transverse distance may then be written:

$$\int_{r_{k-1}}^{r_k} \frac{2\Pi\partial(r)\Pi r^2 dr}{3N_k} = \frac{2\Pi^2}{3N_k}\int_{r_{k-1}}^{r_k} \partial(r)r^2 dr$$

Having to start from the depot at the center of the circular region, go to its assigned zone, do the pick-up and delivery in its zone and come back to the depot in a fixed amount of time, each vehicle in a ring has then to satisfy the following time constraint (there is one constraint per ring):

$$\text{local travel time} + \text{handling time} + \text{line-haul time} \leq T \qquad (1)$$

where T is the time period considered

The **local travel time** (T_1) for a vehicle of the k^{th} ring is the travel time to tour all the pick-up and delivery points of the zone considered. To obtain this, one has to divide by speed v the sum of the transverse and radial displacement times:

$$T_1 = \frac{1}{v} \cdot \left[\frac{2\Pi^2}{3N_k^2} \int_{r_{k-1}}^{r_k} \partial(r) r^2 dr + 2(r_k - r_{k-1}) \right]$$

We call **handling time** (T_2) for a vehicle of the k^{th} ring the time to do the delivery (or the pick-up) at the client's place. To obtain it, the number of points visited in one sector shall be multiplied by the (fixed) handling time at one client's place (t^{liv}):

$$T_2 = \frac{2\Pi}{N_k} \int_{r_{k-1}}^{r_k} \partial(r) r dr \cdot t^{liv}$$

The **line-haul time** (T_3) for a vehicle of the k^{th} ring is the time required to travel (return trip) the distance between the depot and the inside radius of the sector:

$$T_3 = \frac{2r_{k-1}}{v}$$

Equation (1) then becomes:

$$T_1 + T_2 + T_3 \leq T$$

or, for a vehicle of the k^{th} ring:

$$\left[\frac{2\Pi^2}{3N_k^2} \int_{r_{k-1}}^{r_k} \partial(r) r^2 dr + 2(r_k - r_{k-1})\right]/v + \frac{2\Pi t^{liv}}{N_k} \int_{r_{k-1}}^{r_k} \partial(r) r dr + \frac{2r_{k-1}}{v} \leq T \qquad (2)$$

The utilization time of the vehicles depends on the N_k's and the r_k's which have to be determined so as to satisfy the constraint (2) and to minimize the number of vehicles or the total distance traveled by the set of vehicles.

The objective function is:

$$\text{Min } D = \sum_k \left[\frac{2\Pi^2}{3N_k} \int_{r_{k-1}}^{r_k} \partial(r) r^2 dr + 2r_k N_k \right] \qquad (3a)$$

for the total distance or

$$\text{Min } \sum_k N_k \qquad (3b)$$

for the total number of vehicles.

3.2. Resolution

We reformulate the problem in a form easier to solve, then demonstrate the equivalence of objectives (3a) and (3b) and finally we propose the algorithm of solution and some results.

3.2.1. Reformulation for the objective of minimizing the total distance

Let us first consider the problem of minimizing the total distance. The minimization of the total number of vehicles is examined in section 3.2.2. We first show that the constraints (2) are saturated at optimum. To do this, we compare two solutions which differ only by the penultimate radius (r_{k-1}) and demonstrate that the solution with the smallest r_{k-1} dominates the other one.

Thus with the N_k's fixed and $N_k > N_{k-1}$, solution A gives the radii r_k, $r_{k-1}, r_{k-2}, \ldots, r_1$ and solution B the radii $r_k, r'_{k-1}, r_{k-2}, \ldots, r_1$ with $r_{k-1} < r'_{k-1}$. The total distance traveled by the vehicles of the last two rings is:

$$D_A = \frac{2\Pi^2}{3N_k} \int_{r_{k-1}}^{r_k} \partial(r) r^2 dr + 2r_k N_k + \frac{2\Pi^2}{3N_{k-1}} \int_{r_{k-2}}^{r_{k-1}} \partial(r) r^2 dr + 2r_{k-1} N_{k-1}$$

for solution A and

$$D_B = \frac{2\Pi^2}{3N_k}\int_{r'_{k-1}}^{r_k} \partial(r)r^2dr + 2r_kN_k + \frac{2\Pi^2}{3N_{k-1}}\int_{r_{k-2}}^{r'_{k-1}} \partial(r)r^2dr + 2r'_{k-1}N_{k-1}$$

for solution B.

We have $$D_B - D_A = \left(\frac{1}{N_{k-1}} - \frac{1}{N_k}\right)\frac{2\Pi^2}{3}\int_{r_{k-1}}^{r'_{k-1}} \partial(r)r^2dr + 2N_{k-1}(r'_{k-1}-r_{k-1})$$

As $N_k > N_{k-1}$ (hypothesis that may be considered reasonnable since it can be expected that the further a ring is from the center the greater should be the number of vehicles required to cover it; this hypothesis is in any case confirmed in practice - see sub-section 3.2.4), then $D_B - D_A > 0$ and solution A is better. So, for all the fixed N_k's (and particularly for the optimal N_k's), the best solution is obtained by saturating the constraints (2).

In order to simplify objective (3a) and inequation (2), we utilize a linear approximation of the function $\partial(r)$. Let us assume $\partial(r)$ constant over the interval $[r_{k-1}, r_k]$ although varying from one interval to the other. In the k^{th} interval $\partial(r)$ is approximated by $\partial(r_k^*)$

$$\text{where } r_k^* = (r_k + r_{k-1})/2.$$

The objective function (3a) becomes:

$$D = \sum_k \left[\frac{2\Pi\partial(r_k^*)(r_k^3-r_{k-1}^3)}{9N_k} + 2r_kN_k\right] \qquad (4)$$

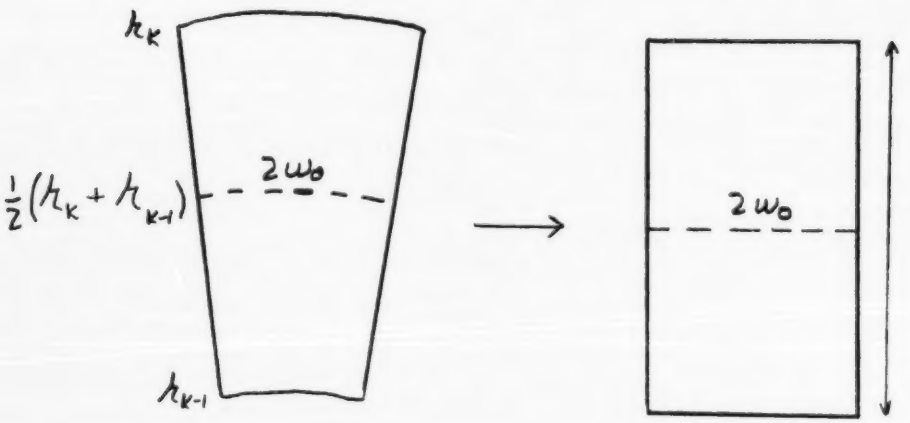
Inequation (2) becomes:

$$\frac{2\Pi^2\partial(r_k^*)(r_k^3-r_{k-1}^3)}{9N_k^2v} + \frac{\Pi t^{liv}\partial(r_k^*)(r_k^2-r_{k-1}^2)}{N_k} + \frac{2r_k}{v} \leq T \qquad (5)$$

In order to solve the mathematical program formed by the objective function (4) and the constraints (5), we will substitute to the objective function a relation which corresponds to the conditions of optimality. These conditions, described by Newell and Daganzo [1986], require the sectors to

have the optimal shape as described in section 2 e.g. that they have a width equal to $2w_0$.

A sector of a ring can be approximated by a rectangle of length $r_k - r_{k-1}$ and of a width equal to the length of the circular arc of radius $(r_k + r_{k-1})/2$.



For the k^{th} ring, the number of zones N_k is given by

$$N_k = \frac{\text{average circumference}}{\text{optimal width of zone}} = \frac{2\Pi(r_k + r_{k-1})/2}{2w_0}$$

As $w_0 = (3/\partial)^{1/2}$,

$$\text{then } N_k = \frac{\Pi(r_k + r_{k-1})\partial^{1/2}}{2\sqrt{3}}. \qquad (6)$$

When N_k is replaced by its value and $\partial(r_k^*)$ is used as the value of ∂, inequation (5) becomes:

$$\frac{8}{3v}\partial(r_k^*)(r_k^3 - r_{k-1}^3) + 2\sqrt{3}\partial^{3/2}(r_k^*)(r_k + r_{k-1})(r_k^2 - r_{k-1}^2)t^{liv} + \frac{2r_k}{v}\partial(r_k^*)(r_k + r_{k-1})^2 \leq \partial(r_k^*)(r_k + r_{k-1})^2 T \qquad (7)$$

3.2.2. Equivalence of the objectives

We will demonstrate that minimizing the number of vehicles is equivalent to minimizing the total distance (for all the rings except possibly the inner one) and that thus the previous method remains valid for either one of the objectives.

For all the rings except the central one, the vehicles must be constantly occupied in order to minimize their number. Let $N' = \sum_{k=1} N_k$. We then have:

$$N' \cdot T = \text{total traveling time} + \text{total handling time}$$
$$= \frac{D'}{v} + K$$

where D' is the total travel for all rings except the inner one and K the (fixed) handling time for all points in these rings.

Thus, since T, v and K are constant and positive, minimizing N' comes to minimizing an increasing linear function of D' and is then equivalent to minimizing D'.

3.2.3. Algorithm of solution

The method will consist essentially in finding successively the r_k's from the outer ring toward the center and each time saturating inequation (7). This approach is so much justified of being used as the contribution from the outer rings, containing more zones, is definitely more important than that of the inner rings.

1. Starting from the outer-most ring (r_k=R=radius of region), we find the r_{k-1} which saturates inequation (7). A method of bi-section could be used for example. And the procedure is repeated for each ring toward the center.

2. Equation (6) allows to find, from the r_k's, the N_k's that are rounded off to the nearest integer.

3. The r_k's may be readjusted by equation (6) by using the integer N_k's found in 2 above.

4. The total distance is calculated

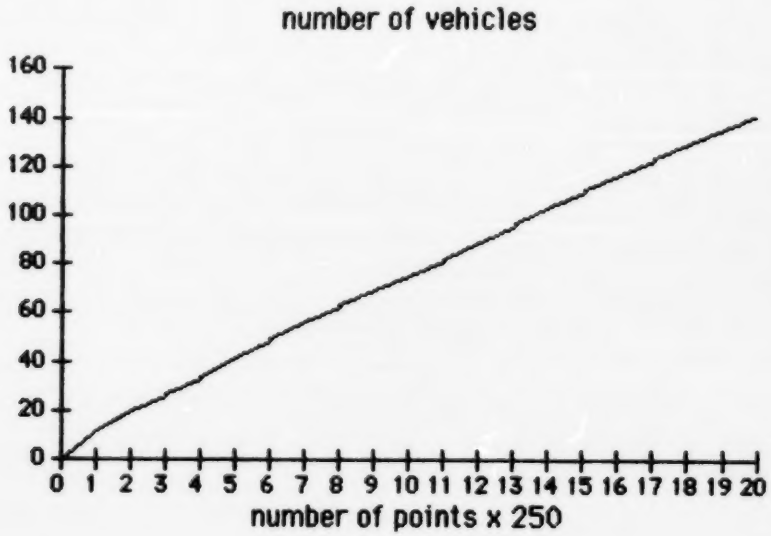
$$D = \sum_k \left[\frac{2\Pi\partial(r_k^*)(r_k^3 - r_{k-1}^3)}{9N_k} + 2r_kN_k \right]$$

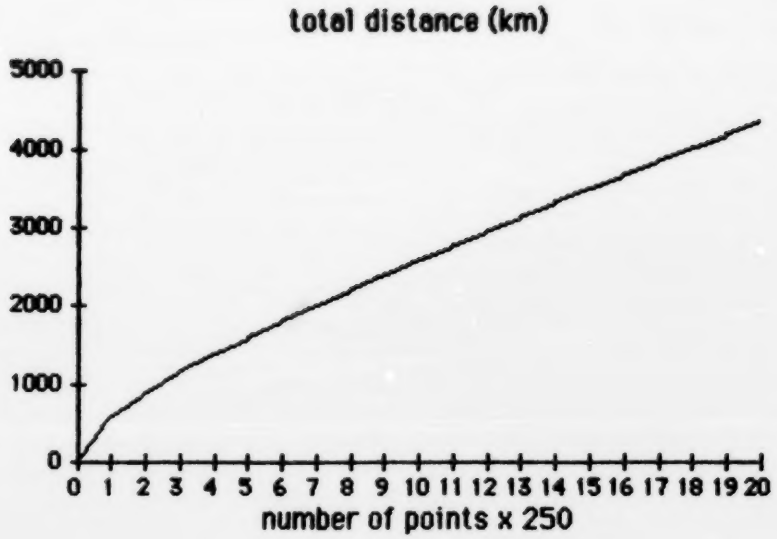
or the total number of vehicles

$$\sum_k N_k$$

3.2.4. Numerical results

The preceding algorithm was used to determine the partitioning of the zones by varying the parameters T, t^{liv} and n. The radius of the region was set at 15 km and the speed of vehicles at 30 km/h. The following two graphs show the results obtained for the number of vehicles and the total distance traveled in relation to the the number of points to be visited with T=4 h and t^{liv}=5 min.





This last graph clearly indicates the ecomomy of scale obtained as the number of points increases. Afterwards, we have determined the partitioning by successively giving the values 4, $2^2/_3$ and 2 h. to the parameter T (planning period), 3 and 5 min. to the parameter t^{liv} (delivery time) and 1000, 2000, 3000, 4000 and 5000 to the parameter n (total number of points).

We have observed that as the delivery time decreases, the shape of the zones stretches in direction of the center. Similarly, the longer the planning period, the more elongated the shape of the zones gets. Thus, the shape is so much more elongated as the number of points to be visited by one vehicle is great. We have also noticed that the form of the zones e.g. the ratio of the length to the width varies very little in relation to the density. The following table shows the results obtained for n=3000:

T	2 h		2.67 h		4 h	
t^{liv}	3 min	5 min	3 min	5 min	3 min	5 min
number of rings	5	8	4	6	3	4
number of vehicles	145	221	99	146	60	89
total distance(km)	4164	5657	3159	4118	2438	2969
average number of parcels/veh.	20.6	13.6	30.4	20.5	49.9	33.6

We can see that as the period increases the number of vehicles, the distance traveled and the number of rings decrease while the average number of parcels per vehicle increases.

The table below shows the detailled results obtained for each ring when t^{liv}=5 and n=3000 (results obtained after steps 1 and 2 of the algorithm e.g. before rounding off the results). It is to be noted that varying the parameter t^{liv} downward is equivalent to varying the parameter T upward.

rings	inside radius(km)	No of veh.	dist./veh.(km)	area/veh.(km^2)	length/width
			T=4		
1	0	1.5	3.2	0.9	-
2	0.7	13.2	22.2	6.8	3.3
3	5.4	30.7	30.2	7.6	3.0
4	10.2	43.9	39.8	8.7	2.7
			T=2.67		
1	0	0.7	1.4	0.4	-
2	0.3	8.2	14.4	4.4	2.2
3	3.4	20.6	19.2	4.6	2.0
4	6.4	31.1	24.8	4.7	1.8
5	9.4	39.6	30.2	4.9	1.7
6	12.2	46.1	35.5	5.1	1.5
			T=2		
1	0	2.4	5.0	1.5	-
2	1.1	9.8	11.7	3.3	1.6
3	3.4	18.9	15.6	3.3	1.5
4	5.6	26.7	19.6	3.3	1.3
5	7.7	33.4	23.3	3.2	1.2
6	9.7	39.0	26.9	3.2	1.1
7	11.5	43.5	30.2	3.2	1.0
8	13.3	47.1	33.4	3.2	0.9

It can be seen that the number of vehicles per ring and the distance traveled increase from the center toward the periphery. The fact that the area covered by each vehicle also increases is explained by the density which decreases toward the periphery. It should be noted that the areas become more and more square as we are getting further from the center.

We have finally compared the results obtained when visiting 4000 clients in 4 hours with those obtained when visiting twice 2000 clients in 2

hours, which corresponds to switching from a delivery service in one day to a delivery service in half a day.

	total distance (km)	No of vehicles	dist./veh.(km)	# parcels/veh.
4000 clients in 4h	3680	117	31.4	34
2x(2000 clients in 2h)	8118	152	53.2	26

This change in the service carries a 30% increase in the number of vehicles required. This increase remains the same when other values of the different parameters are used.

In the following section, we will examine more closely the design of the zones for the inner-most rings.

3.3. The inside rings

The previous method is supported by the fact that the zones are almost rectangular and that for a rectangular zone it is possible to determine the width that minimizes the total distance traveled in the zone. Nevertheless, for the rings closest to the center, this assumption of rectangularity is not maintained and, particularly for the inner-most ring, the number of vehicles is certainly not optimal.

Therefore it is necessary to reconsider the distribution of the zones in the two central rings. We will at the same time, whenever there are two rings, examine the possibility to move the boundary r_1 between both zones.

3.3.1. Case 1: one ring only (radius=R)

The problem to be solved can be expressed as:

$$\text{Min } D = \frac{2\Pi^2}{3N}\int_0^R \partial(r) r^2 dr + 2RN$$

or

$$\text{Min } N \quad \text{(number of vehicles)} \tag{8}$$

subject to

$$\frac{2\Pi^2}{3N^2 v}\int_0^R \partial(r) r^2 dr + \frac{2R}{v} + \frac{2\Pi t^{liv}}{N}\int_0^R \partial(r) r dr \leq T$$

N positive integer

In the case where it is wanted to minimize N, the problem to be solved may be written with a simplified notation:

$$\text{Min } N$$

$$\text{s. t.}$$

$$\frac{c}{N^2} + \frac{d}{N} \leq e$$

$$N \text{ positive integer}$$

The unique constraint may be expressed as a quadratic inequation:

$$eN^2 - dN - c \geq 0$$

with

$$c = \frac{2\Pi^2}{3v} \int_0^R \partial(r) r^2 dr > 0$$

$$\text{and} \quad e = T - \frac{2R}{v} > 0$$

(in one period T, a vehicle must have time to make a return trip from the depot to the periphery in order for the problem to have a solution).

Since $c>0$ and $e>0$, the two roots ρ_1 et ρ_2 of equation $eN^2 - dN - c = 0$ are of different signs ($\rho_2>0$) and the solution set is limited to $[\rho_2, \infty[$ because N has to be positive. We can then replace the inequation by

$$N \geq \rho_2 = \frac{d + \sqrt{d^2 + 4ec}}{2e}$$

Therefore it is sufficient in order to solve the previous mathematical problem to take the positive root ρ_2 as N^* and to round off to the integer greater than or equal to N^*.

In the case where we want to minimize the total distance, the problem may be written under the following simplified form:

$$\text{Min}_{N} \; \frac{a}{N} + bN$$

s. t.

$$N \geq \rho_2$$

$$N \text{ integer}$$

The minimum of the objective function, if the constraint is ignored, is obtained for $N^* = (a/b)^{1/2}$.

If N^* is feasible, then the only three candidate-solutions to be considered are the rounded up and rounded down N^* and the rounded up ρ_2, the optimal solution being the best one of the three.

3.3.2. Case 2: more than one ring

The vehicles of the first ring not being used at 100% in the period T, we will reconsider the distribution in the two inner-most rings, the 2nd having r_1 as inside radius and r_2 as outside one (r_2 fixed). We will also reconsider the position of boundary r_1 between the first two rings in such a way as to re-optimize the total travel time in them.

The problem is then the following:

$$\text{Min } D = \frac{2\Pi^2}{3N_1}\int_0^{r_1} \partial(r)r^2dr + 2r_1N_1 + \frac{2\Pi^2}{3N_2}\int_{r_1}^{r_2} \partial(r)r^2dr + 2r_2N_2$$

or

$$\text{Min } N_1 + N_2 \quad \text{(number of vehicles)}$$

subject to

$$\frac{2\Pi^2}{3N_1{}^2v}\int_0^{r_1} \partial(r)r^2dr + \frac{2r_1}{v} + \frac{2\Pi t^{liv}}{N_1}\int_0^{r_1} \partial(r)rdr \leq T$$

$$\frac{2\Pi^2}{3N_2{}^2v}\int_{r_1}^{r_2} \partial(r)r^2dr + \frac{2r_2}{v} + \frac{2\Pi t^{liv}}{N_2}\int_{r_1}^{r_2} \partial(r)rdr \leq T$$

$$0 \leq r_1 \leq r_2$$

N_1, N_2 positive integers

Method of solution:

Letting r_1 vary from 0 to r_2, N_1 and N_2 are calculated for each value of r_1 by applying separately the previous method to each ring. It should be noted that in the case of the 2nd ring, we obtain the same problem as the preceding one (8), except that the limits of integration are r_1 and r_2 instead of 0 and R.

To summarize, the assignment of zones is made in two steps: first, the assignment is made from the periphery toward the center, saturating constraint (7) for each ring. Then, as this method is not giving good results near the center, the design of the zones in the two central rings is reconsidered by the method just described.

4. Variability of the demand

The preceding analysis determines for a certain demand n the **mean** number n_k of points to be visited in each zone of the k^{th} ring. Once the zones are determined, the number of points to be visited per period T in a zone of the k^{th} ring is distributed according to a binomial law $b(n;p=n_k/n)$ which can be approximated by a normal law of mean $\mu_k = n_k$ and variance $\sigma_k^2 = n_k(1-n_k/n)$.

Thus, since the zones are determined in relation to the mean numbers, it can be expected that in 50% of the cases, the number of points to be visited will be greater than n_k and that the vehicle will not have the time to visit all of them. To overcome this difficulty, we could determine the zones in such a way that in 85% of the cases for instance the vehicle has the time to visit them all.

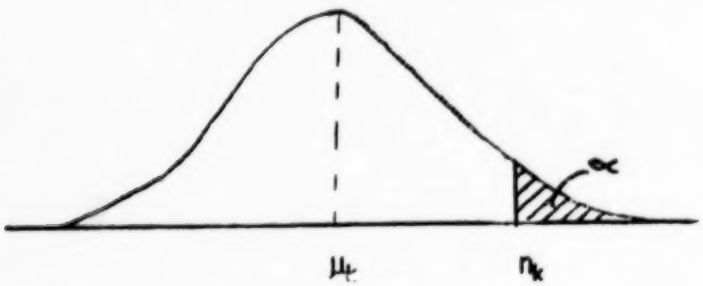
4.1. Planning without zone overlapping

We want to determine smaller zone so that the vehicles have time to visit all the clients in their zone in 85% of the cases. If n is the number of pick-up and delivery points to be visited, we plan the partitioning of the

region with $n^+ > n$ in order to obtain smaller zones. An iterative procedure is used to determine the adequate n^+:

1- We choose one $n^+ > n$ from which can be determined by the method of section 3 the optimal zoning e.g. the number N_k of zones and the number r_k of points per zone in the k^{th} ring.

2 - We want to determine the μ_k such that $P[X>r_k] \leq \alpha$ with $X \sim N(\mu_k,\sigma_k)$. Let z_α be such that $P[z>z_\alpha] = \alpha$ with $z \sim N(0,1)$;



μ_k may be determined by solving the equation

$$\mu_k + z_\alpha\sqrt{\mu_k(1-\mu_k/n^+)} = r_k$$

which can be transformed into the quadratic equation:

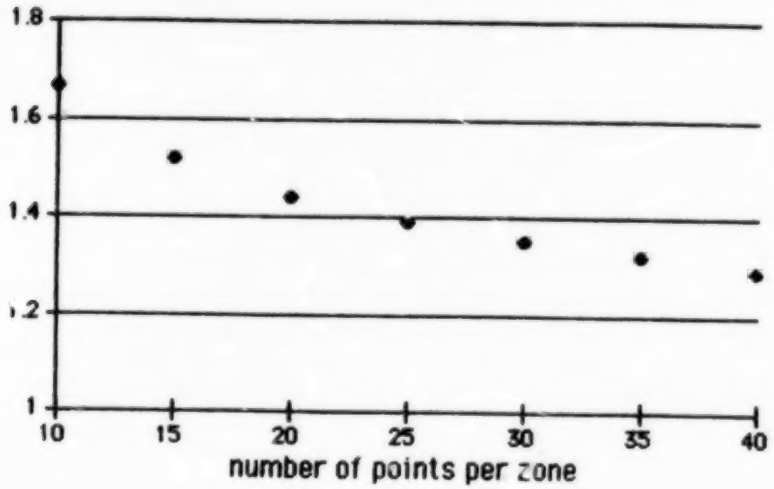
$$z_\alpha^2\mu_k(1-\mu_k/n^+) = (r_k - \mu_k)^2$$

3 - From the μ_k's, we can establish the total number of demands for the whole region:

$$n' = \Sigma_k \mu_k N_k$$

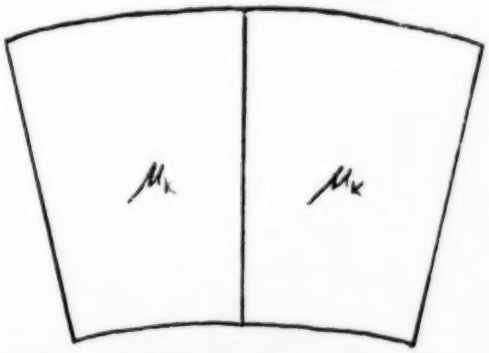
4 - If n' is not close enough to the searched n, we start over again at step 1 by increasing or decreasing n^+.

The following graph gives the ratio n_k/μ_k for various values of n_k. We have found the same ratios whatever was the total number of demands n. The graph then illustrates the cost increases (in terms of vehicles required) to take into account the daily variation.



4.2. Planning with zone overlapping

To study the possible overlapping of the zones, we will group the zones by pairs so as to reduce the variance and thus reduce the number of vehicles required to obtain a sufficient coverage in x% of the cases.



Let μ_k be the mean number of points in a zone of the k^{th} ring. The variance is then given by

$$\sigma_k^2 = \mu_k(1-\mu_k/n).$$

If X_k is the random variable of the number of points in two adjacent zones of the k^{th} ring, we have:

$$\mu(X_k) = 2\mu_k$$
$$\text{var}(X_k) = 2\mu_k(1-2\mu_k/n)$$

and if we divide by 2 (for the two vehicles) the number of points, we obtain

$$\mu(X_k/2) = \mu_k$$
$$\text{var}(X_k/2) = (1/2)\mu_k(1-2\mu_k/n)$$

Generally, if we group together m adjacent zones and allow the redistribution of the points to be visited between the m vehicles, the mean and variance of the number of points visited by each vehicle are:

$$\mu_m = \mu_k$$
$$\sigma_m^2 = (1/m)\mu_k(1-m\mu_k/n)$$

Thus for a coverage at level α the number of points per zone is

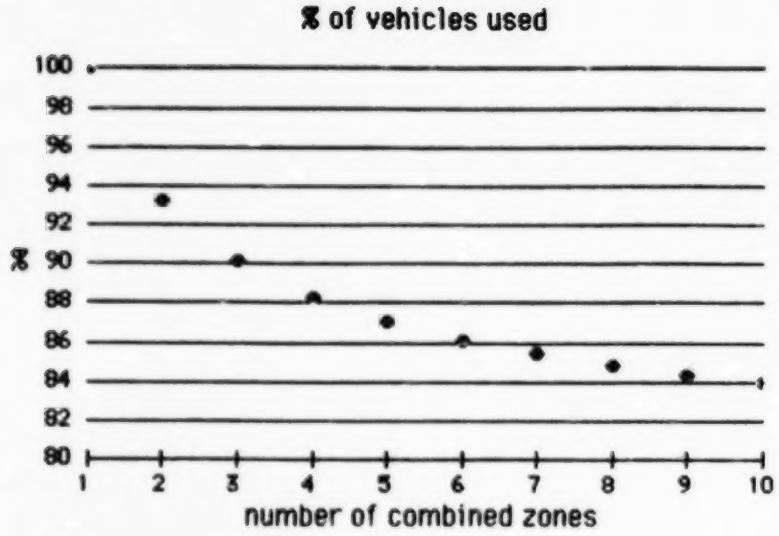
$$n_k = \mu_k + z_\alpha\sqrt{\mu_k(1-\mu_k/n)}$$

when there is no overlapping and

$$n_k' = \mu_k + z_\alpha\sqrt{(1/m)\mu_k(1-m\mu_k/n)}$$

when there is overlapping of m zones.

The gain on the number of vehicles when overlapping of 2 zones is permitted varies from 6 to 9% according to n_k, the number of points per zone and n, the total number of points. The gain due to the addition of each supplementary overlapping zone decreases with the number of zones combined. The following graph indicates the % of the vehicles used in relation to the initial number (without overlap) when we allow the overlapping of zones for n=3000 and n_k=30.



We can see that there is a trade-off between the gain to be obtained and the difficulty of routing. In fact, the more the overlapping is permitted, the more difficult become the problem to solve and the organization to manage.

5. Conclusion

We have developed an approximate analytical method of planning the pick-up and delivery zones assigned to the vehicles of a fleet. This method is simple to use and permits to evaluate the feasibility of different organization schemes for a delivery system. Furthermore we have adapted the method so that the daily variability of the demand may be taken care of. Finally, in the last section, we have estimated the possible gain on the number of vehicles used if zone overlapping is permitted in order to reduce the variance of the demand.

The utility of the method is related to the importance of a good strategic planning. The choice of the form of the most efficient organization may, in our opinion, lead to gains in productivity more significant than those permitted by tactical decisions (the day-by-day routing, for instance).

Moreover, the results obtained by our method are, despite the approximations, more representative of the average performance over a long period of time than the ones of a tactical model. And, another advantage, the development, calibration and utilization of such a model are certainly less

expensive and faster to finalize, particularly if it is required to test several alternatives of organization.

At the tactical level, as methods of routing use mainly the cluster-first approach, the shape of the zones determined by our method can initiate this clustering and therefore lead to good solutions since the clustering is done with a global perpective. Once the clustering is done, the routing of each vehicle can be carried out using approximate methods (the strategy of Daganzo [1984a] for instance) or exact methods. For a review of the routing algorithms, the reader should refer to Bodin et al.[1983], Laporte and Nobert[1985] and Langevin [1986].

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